Your Name

Your Signature

Student ID #

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. Make sure that it is easy for graders to follow what you are doing [e.g. if you perform more than one row operation in a step, label what you've done. If you perform just one row operation at a time, it should be clear what you're doing, and so the row operation does not need to be labeled in that case].
- Place a box around your answer to each question.
- **You may write on the backs of pages** (and are expected to for some questions, in order to have enough space). Both sides of each page of the exam will be scanned.
- Raise your hand if you have a question.
- This exam has 5 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	15	
3	14	
4	10	
Total	50	

1. (11 points) **For this problem only, you do not have to show work.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

T	$\boxed{F}$	If $\text{span}(\mathbf{v_1}, \mathbf{v_2}) = \text{span}(\mathbf{u_1}, \mathbf{u_2})$, then $\mathbf{v_1} = \mathbf{u_1}$ and $\mathbf{v_2} = \mathbf{u_2}$.
$\boxed{T}$	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans $\mathbb{R}^n$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ spans $\mathbb{R}^n$.
T	$\boxed{F}$	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly independent, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly independent.
$\boxed{T}$	F	Suppose that one can go from the system $[A\|\mathbf{b}]$ to the system $[A'\|\mathbf{b}']$ using row operations. Then, both systems have the same solution sets.
$\boxed{T}$	F	Any linear system of equations has either 0, 1, or infinitely many solutions.
$\boxed{T}$	F	If $\mathbf{v_1} \in \text{span}(\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly dependent.
T	$\boxed{F}$	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ span $\mathbb{R}^3$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly dependent.
$\boxed{T}$	F	If $[A\|\mathbf{b}]$ has infinitely many solutions, then so does $[A\|\mathbf{0}]$.
$\boxed{T}$	F	If a linear system of equations has more variables than it has equations, then it has either 0 or infinitely many solutions.
T	$\boxed{F}$	Any list containing $\mathbf{0}$ is linearly independent.
T	$\boxed{F}$	If A is a 5×3 matrix and $\text{REF}(A)$ has a pivot in every column, then for any $\mathbf{b} \in \mathbb{R}^5$, $[A\|\mathbf{b}]$ has at least one solution.

2. (15 points) Note: in this problem, you should be able to reuse your computation from part (a) to do all the other parts. Consider the vectors

$$\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u_4} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}.$$

(a) Compute the reduced echelon form of the augmented matrix $[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\ \mathbf{u_4}|\mathbf{b}]$.

(b) Give the solution set to the matrix equation $[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\ \mathbf{u_4}]\mathbf{x} = \mathbf{b}$ in parametric form.

(c) Write $\mathbf{b}$ as a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$ (i.e. give the explicit scalar coefficients).

(d) Is $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ a linearly independent list? Briefly explain how you know.

(e) Does the list $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ span $\mathbb{R}^3$? Briefly explain how you know.

(f) Is $\mathbf{u_4} \in \text{span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$? Briefly explain how you know.

Solution.

(a) We perform row operations to compute

$$\left[\begin{array}{cccc|c} 0 & 2 & -1 & 1 & 3 \\ 1 & 0 & 2 & -1 & 2 \\ -1 & -2 & -1 & 0 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 & 3 \\ -1 & -2 & -1 & 0 & -5 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 & 3 \\ 0 & -2 & 1 & -1 & -3 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(b) The parametric form of the solution set is

$$\left\{ \begin{bmatrix} 2 \\ \frac{3}{2} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

(c) Setting s and t to 0 in the parametric form, we see that a solution is given by $x_1 = 2, x_2 = \frac{3}{2}, x_3 = 0, x_4 = 0$, so we can write

$$2\mathbf{u_1} + \frac{3}{2}\mathbf{u_2} = \mathbf{b}.$$

(d) The computation above shows that the reduced echelon form of $[\mathbf{u_1}\mathbf{u_2}\mathbf{u_3}\mathbf{u_4}]$ is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which does not have a pivot in every column, so $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ is *not* a linearly independent list.

(e) The reduced echelon form of $[\mathbf{u_1}\mathbf{u_2}\mathbf{u_3}\mathbf{u_4}]$ does not have a pivot in every row, so $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ does *not* span $\mathbb{R}^3$.

(f) Doing the same row operations as in part (a) shows that the reduced echelon form of $[\mathbf{u_1}\mathbf{u_2}\mathbf{u_3}|\mathbf{u_4}]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which shows that this system is consistent (the REF has no row of the form $[000| \neq 0]$). Therefore $\mathbf{u_4} \in \operatorname{span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$.

3. (14 points) Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & a & -1 & 0 \\ 0 & 3 & b & 0 & 0 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & d & 0 & 2 \end{array}\right].$$

For each of the following parts, **either** give an example of a specific choice of $(a,b,c,d) \in \mathbb{R}^4$ such that the given property holds **or** briefly explain why it is impossible to find such a choice of $(a,b,c,d) \in \mathbb{R}^4$.

(a) the given augmented matrix is *not* in echelon form

Solution. e.g. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) the given augmented matrix is in *reduced* echelon form

Solution. This is impossible, since (e.g.) no matter how we modify the third column, the second row will have a pivot which is equal to 3.

(c) the given augmented matrix is in echelon form and has a pivot in its third column

Solution. e.g. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

(d) the given augmented matrix is in echelon form and has *no* pivot in its third column

Solution. e.g. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(e) the given augmented matrix is in echelon form and the associated linear system has infinitely many solutions

Solution. This is impossible. For the augmented matrix to be in echelon form, we must have that $d = 0$ (if the index of the leading nonzero entry is strictly increasing from row to row, the first nonzero entry of the fourth row must be in the fourth column or later). But whenever $d = 0$, the last row encodes the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 2$, so the associated system is inconsistent and has *no* solution. Thus, no matter what the third column is, the system cannot be both in echelon form and have infinitely many solutions.

(f) the given augmented matrix is in echelon form and the associated linear system has no solution

Solution. e.g. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(g) the linear system associated to the given augmented matrix has exactly one solution

Solution. e.g. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. (You didn't have to explain why your solution works, but to see that it does, we can switch the 3rd and 4th rows to put the system into echelon form, and then we can see that the system is consistent and has no free variables, and therefore has exactly one solution.)

4. (10 points) Note: for this question, row operations will not be particularly helpful. Consider some vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{b} \in \mathbb{R}^n$. Suppose that we know that

$$2\mathbf{v_1} + \mathbf{v_4} - \mathbf{v_3} = \mathbf{b}.$$

(a) Give a solution to the matrix equation $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}]\mathbf{x} = \mathbf{b}$.

Solution. $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(b) Is $\mathbf{v_3} \in \text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}, \mathbf{b})$? Explain your answer.

Solution. We can add $\mathbf{v_2} - \mathbf{b}$ to both sides of the given vector equation to get

$$2\mathbf{v_1} + \mathbf{v_4} - \mathbf{b} = \mathbf{v_3},$$

so $\mathbf{v_3}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_4}, \mathbf{b}$ and so $\mathbf{v_3} \in \text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}, \mathbf{b})$.

(c) Suppose that we *also* know that

$$\mathbf{v_1} + \mathbf{v_2} = \frac{1}{2}\mathbf{b}.$$

Show that the list $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly dependent. (Hint: think about different ways you could combine our vector equations, and which definition of linear dependence is most convenient to use here.)

Solution. Multiplying the given equation by 2 on both sides gives

$$2\mathbf{v_1} + 2\mathbf{v_2} = \mathbf{b}.$$

The first given vector equation also expresses $\mathbf{b}$ as a linear combination, and setting these two expressions for $\mathbf{b}$ equal gives

$$2\mathbf{v_1} + \mathbf{v_4} - \mathbf{v_3} = 2\mathbf{v_1} + 2\mathbf{v_2}.$$

Subtracting $2\mathbf{v_1} + 2\mathbf{v_2}$ from both sides of this equation gives

$$0\mathbf{v_1} + (-2)\mathbf{v_2} + (-1)\mathbf{v_3} + (1)\mathbf{v_4} = \mathbf{0}.$$

Therefore, there is a nontrivial linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ which is equal to the zero vector, which means that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is a linearly *dependent* list.

(d) (For this part, continue to assume that the equation in part (c) holds.) How many solutions does the system $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4} |\ 2\mathbf{b}]$ have? Explain how you know.

Solution. First, the system $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4} |\ 2\mathbf{b}]$ is consistent because, for instance, $4\mathbf{v_1} + 4\mathbf{v_2} + 0\mathbf{v_3} + 0\mathbf{v_4} = 2\mathbf{b}$, as can be seen by multiplying both sides of the equation given in (c) by 4. Moreover, since we established in part (c) that the columns $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ are linearly dependent, the system has free variables; thus the system has infinitely many solutions.

Alternate solution. Multiplying the given equations on both sides by scalars shows that

$$4\mathbf{v_1} + 0\mathbf{v_2} + (-2)\mathbf{v_3} + 2\mathbf{v_4} = 2\mathbf{b}$$

$$4\mathbf{v_1} + 4\mathbf{v_2} + 0\mathbf{v_3} + 0\mathbf{v_4} = 2\mathbf{b},$$

so the system has at least two solutions, $\begin{bmatrix} 4 \\ 0 \\ -2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$. But if a linear system has more than one solution, it must have infinitely many; so the system has infinitely many solutions.